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A PACKET OF ELASTIC PLATES JOINED ALONG A PERIODIC SYSTEM OF SEGMENTS

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A packet of infinite thin elastic plates joined along a periodic system of collinear segments is considered. The elastic properties and thicknesses of the plates can be different. The plates are loaded by tensile forces at infinity. An algorithm for determining the complex potentials governing stresses in the plates is constructed by solving the Riemann matrix boundary-value problem. The stress-intensity factors are found and their graphs are given.

Formulation of the Problem. Let n infinite thin elastic homogeneous isotropic plates $E_1, E_2, \ldots, E_n$ occupying the entire complex plane $z = x + iy$ be superimposed on one another and joined along a periodic system of segments $l_j = [a + jT, b + jT]$ $(j = 0, \pm 1, \pm 2, \ldots)$ on the real x axis. It is assumed that there are no interlayers and initial tension between the plates. The plate E_k $(k = \overline{1, n})$ has thickness h_k, shear modulus μ_k, and Poisson's ratio ν_k. In the band of periods in the plate E_k, the stresses $(\sigma_x^\infty)'_k$, $(\sigma_y^\infty)'_k$, and $(\tau_{xy}^\infty)'_k$ normalized by the plate thickness and the rotation $(\omega^\infty)'_k$ are specified for $y \to +\infty$, and the corresponding quantities $(\sigma_x^\infty)''_k$, $(\sigma_y^\infty)''_k$, $(\tau_{xy}^\infty)''_k$, and $(\omega^\infty)''_k$ are specified for $y \to -\infty$.

The following assumptions are used: 1) the plates are in a generalized plane stress state and interact only along the joint lines, the spatial effect of stress concentration is negligible on the joint lines, and friction between the plates is absent; 2) at the ends of the segments l_j, the stresses and derivatives of displacements with respect to x can tend to infinity with an order less than 1 (at the remaining points, these quantities are continuous).

On the joint lines, the following conjugation conditions must be satisfied:

$$(u + iv)_k^+ = (u + iv)_k^-, \quad k = \overline{1, n}, \qquad (u + iv)_k^+ = (u + iv)_{k+1}^+, \quad k = \overline{1, n-1},$$
$$\sum_{k=1}^n h_k(\sigma_y - i\tau_{xy})_k^+ = \sum_{k=1}^n h_k(\sigma_y - i\tau_{xy})_k^-. \tag{1.1}$$

Here $(u + iv)_k$ is the displacement vector of the plate E_k and $(\sigma_y, \tau_{xy})_k$ are the normal and shear stresses in the plate E_k, respectively, normalized by the plate thickness. The first $2n - 1$ conditions in (1.1) express the equality of the displacements of the plates $E_1, E_2, \ldots, E_n$ on the joint line and the last condition is the condition of equilibrium on this line.

It is required to determine the periodic stress state of the packet of plates described above. For two plates, this problem was solved in [1]. Packets of plates joined along a finite number of segments, concentric circles or open curves were studied in [2–4].

For the plate E_k, we express the stresses, rotation, and partial derivatives of the displacements with respect to x in terms of two piecewise holomorphic functions $\Phi_k(z)$ and $\Omega_k(z)$ [5]:

$$(\sigma_x + \sigma_y)_k = 4 \operatorname{Re} \Phi_k(z), \quad 2\mu_k \omega_k = (1 + æ_k) \operatorname{Im} \Phi_k(z), \quad æ_k = (3 - \nu_k)/(1 + \nu_k),$$
$$(\sigma_y - i\tau_{xy})_k = \Phi_k(z) + \Omega_k(\bar z) + (z - \bar z)\overline{\Phi'_k(z)}, \tag{1.2}$$
$$2\mu_k(u' + iv')_k = æ_k \Phi_k(z) - \Omega_k(\bar z) - (z - \bar z)\overline{\Phi'_k(z)}, \qquad k = \overline{1, n}.$$

The functions $\Phi_k(z)$ and $\Omega_k(z)$ are periodic with fundamental period T. According to the theory of periodic analytic functions [6], in the band of periods $0 \leqslant \operatorname{Re} z \leqslant T$ at infinity, these functions are given by

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$$\Phi_k(z) = \gamma_k' + O(\mathrm{e}^{-|y|}), \quad \Omega_k(z) = \delta_k' + O(\mathrm{e}^{-|y|}) \quad \text{as} \quad y \to +\infty; \tag{1.3}$$

$$\Phi_k(z) = \gamma_k'' + O(\mathrm{e}^{-|y|}), \quad \Omega_k(z) = \delta_k'' + O(\mathrm{e}^{-|y|}) \quad \text{as} \quad y \to -\infty. \tag{1.4}$$

Formulas (1.2), (1.3), and (1.4) can be combined to give

$$\begin{gathered}
\gamma_k' = \frac{1}{4}\,[(\sigma_x^\infty)_k' + (\sigma_y^\infty)_k'] + \frac{2i\mu_k}{1+æ_k}\,(\omega^\infty)_k', \\
\delta_k' = \frac{1}{4}\,[3(\sigma_y^\infty)_k'' - (\sigma_x^\infty)_k''] - i\Big[(\tau_{xy}^\infty)_k'' + \frac{2\mu_k}{1+æ_k}\,(\omega^\infty)_k''\Big], \\
\gamma_k'' = \frac{1}{4}\,[(\sigma_x^\infty)_k'' + (\sigma_y^\infty)_k''] + \frac{2i\mu_k}{1+æ_k}(\omega^\infty)_k'', \\
\delta_k'' = \frac{1}{4}\,[3(\sigma_y^\infty)_k' - (\sigma_x^\infty)_k'] - i\Big[(\tau_{xy}^\infty)_k' + \frac{2\mu_k}{1+æ_k}\,(\omega^\infty)_k'\Big].
\end{gathered} \tag{1.5}$$

2. Solution of the Problem. Using formulas (1.2) and the conjugation conditions (1.1), we obtain the following boundary-value problem for $\Phi_k(z)$ and $\Omega_k(z)$ ($k = \overline{1,n}$) in the class of periodic functions with fundamental period T:

$$\begin{gathered}
æ_k\Phi_k^+(t) - \Omega_k^-(t) = æ_k\Phi_k^-(t) - \Omega_k^+(t), \qquad k = \overline{1,n}, \\
\frac{\mu_{k+1}}{\mu_k}\,[æ_k\Phi_k^+(t) - \Omega_k^-(t)] = æ_{k+1}\Phi_{k+1}^+(t) - \Omega_{k+1}^-(t), \qquad k = \overline{1,n-1}, \\
\sum_{k=1}^n h_k(\Phi_k^+(t) + \Omega_k^-(t)) = \sum_{k=1}^n h_k(\Phi_k^-(t) + \Omega_k^+(t)), \qquad t \in l_0.
\end{gathered} \tag{2.1}$$

Problem (2.1) is written in matrix form

$$A\Phi^+(t) = B\Phi^-(t) \quad \text{or} \quad \Phi^+(t) = A^{-1}B\Phi^-(t), \quad t \in l_0, \tag{2.2}$$

where

$$\Phi(z) = \{\Phi_1, \Phi_2, \ldots, \Phi_n, \Omega_1, \Omega_2, \ldots, \Omega_n\}^{\mathrm{t}},$$

$$A = \begin{pmatrix} A_1 & E \\ A_2 & A_3 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 & E \\ B_2 & B_3 \end{pmatrix},$$

$$A_1 = \operatorname{diag}\{æ_1, æ_2, \ldots, æ_n\}, \qquad B_1 = A_1, \qquad B_2 = -A_3,$$

$$A_2 = \begin{pmatrix} \mu_1^*æ_1 & -æ_2 & 0 & \ldots & 0 & 0 \\ 0 & \mu_2^*æ_2 & -æ_3 & \ldots & 0 & 0 \\ \hdotsfor{6} \\ 0 & 0 & 0 & \ldots & \mu_{n-1}^*æ_{n-1} & -k_n \\ h_1 & h_2 & h_3 & \ldots & h_{n-1} & h_n \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \hdotsfor{4} \\ 0 & 0 & \ldots & 0 \\ -h_1 & -h_2 & \ldots & -h_n \end{pmatrix}, \quad B_3 = \begin{pmatrix} \mu_1^* & -1 & 0 & \ldots & 0 & 0 \\ 0 & \mu_2^* & -1 & \ldots & 0 & 0 \\ \hdotsfor{6} \\ 0 & 0 & 0 & \ldots & \mu_{n-1}^* & -1 \\ -h_1 & -h_2 & -h_3 & \ldots & -h_{n-1} & -h_n \end{pmatrix},$$

$$\mu_k^* = \mu_{k+1}/\mu_k.$$

We seek the eigenvalues of the matrix $A^{-1}B$. To this end, we consider the characteristic equation $|A^{-1}B - \lambda E| = 0$, which can be written as $|B - \lambda A| = 0$. After transformations of the matrix $B - \lambda A$, the equation becomes $C(1-\lambda)^{n+1}(1+\lambda)^{n-1} = 0$, where $C = \text{const} \neq 0$.

One can show that the algebraic multiplicities of the eigenvalues of the matrix $A^{-1}B$ are equal to their geometrical multiplicities. Consequently, the eigenvectors of the matrix $A^{-1}B$ are linearly independent and the matrix $S^{-1}A^{-1}BS$ (S is a matrix whose columns are the eigenvectors of the matrix $A^{-1}B$) is a diagonal matrix with the elements $\lambda_1 = \lambda_2 = \ldots = \lambda_{n+1} = 1$ and $\lambda_{n+2} = \lambda_{n+3} = \ldots = \lambda_{2n} = -1$ [7]. Problem (2.2) is then split into $2n$ independent problems

$$F_j^+(t) = F_j^-(t) \quad (t \in L, \quad j = \overline{1, n+1}), \qquad F_j^+(t) = -F_j^-(t) \quad (t \in L, \quad j = \overline{n+2, 2n}) \tag{2.3}$$

for the components $F_1, F_2, \ldots, F_{2n}$ of the new piecewise holomorphic vector function $F(z) = S^{-1}\Phi(z)$. At the ends of the segment l_0, the function $F(z)$ can tend to infinity with an order less than 1, and in the band of periods at infinity, it has, by virtue of (1.3), (1.4), and (2.2), the representations

$$F(z) = S^{-1}G' + O(\mathrm{e}^{-|y|}) \ \text{ as } \ y \to +\infty, \quad F(z) = S^{-1}G'' + O(\mathrm{e}^{-|y|}) \ \text{ as } \ y \to -\infty; \tag{2.4}$$

$$G' = \{\gamma_1', \gamma_2', \ldots, \gamma_n', \delta_1', \delta_2', \ldots, \delta_n'\}^{\mathrm{t}}, \qquad G'' = \{\gamma_1'', \gamma_2'', \ldots, \gamma_n'', \delta_1'', \delta_2'', \ldots, \delta_n''\}^{\mathrm{t}}, \tag{2.5}$$

where γ_k', γ_k'', δ_k', and δ_k'' are determined by formulas (1.5) and $O(\mathrm{e}^{-|y|})$ is a vector function each component of which is comparable to $\mathrm{e}^{-|y|}$ for large y.

According to [8], problems (2.3) and (2.4) have periodic solutions

$$F_j(z) = d_j, \qquad j = \overline{1, n+1},$$

$$F_j(z) = \chi(z)\Big(c_{0j} + ic_{1j} \cot \frac{\pi z}{T}\Big), \qquad j = \overline{n+2, 2n}, \tag{2.6}$$

$$\chi(z) = \Big(\sin \frac{\pi z}{T}\Big) \Big/ \sqrt{\sin \frac{\pi(z-a)}{T} \sin \frac{\pi(z-b)}{T}};$$

$$(S^{-1}G')_j = (S^{-1}G'')_j = d_j, \qquad j = \overline{1, n+1}; \tag{2.7}$$

$$(S^{-1}G')_j = c_{0j} + c_{1j}, \qquad (S^{-1}G'')_j = c_{0j} - c_{1j}, \qquad j = \overline{n+2, 2n}, \tag{2.8}$$

where $(S^{-1}G')_j$ and $(S^{-1}G'')_j$ are the jth components of the vectors $S^{-1}G'$ and $S^{-1}G''$, respectively, and the function $\chi(z)$ means the single-valued branch in the band $0 \leqslant \operatorname{Re} z \leqslant T$ with a cut along the segment $[a, b]$, whose values tend to 1 as $y \to \pm\infty$. The coefficients c_{0j} and c_{1j} are uniquely determined from (2.8), and equalities (2.7) impose $n+1$ complex conditions on the stresses and rotations in the band of periods at infinity and on the characteristics of the plates. With allowance for (1.5) and (2.5), from equalities (2.7) (as in the case of two plates [1]), we obtain

$$\begin{gathered}(1 + æ_k)[(\sigma_x^\infty)_k' - (\sigma_x^\infty)_k''] = (3 - æ_k)[(\sigma_y^\infty)_k' - (\sigma_y^\infty)_k''], \qquad k = \overline{1, n},\\ (\tau_{xy}^\infty)_k' - (\tau_{xy}^\infty)_k'' = -2\mu_k[(\omega^\infty)_k' - (\omega^\infty)_k''], \qquad k = \overline{1, n},\\ \sum_{k=1}^{n} h_k[(\tau_{xy}^\infty)_k' - (\tau_{xy}^\infty)_k''] = 0, \qquad \sum_{k=1}^{n} h_k[(\sigma_y^\infty)_k' - (\sigma_y^\infty)_k''] = 0.\end{gathered} \tag{2.9}$$

Below, we assume that these conditions are satisfied.

3. Stress-Intensity Factors (SIF). Since $\Phi(z) = SF(z)$, it follows from (2.6) that in the neighborhood of the point $z = b$, the functions $\Phi_k(z)$ and $\Omega_k(z)$ have the form

$$\begin{gathered}\Phi_k(z) = A_k(z-b)^{-1/2} + O(1), \qquad \Omega_k(z) = -æ_k A_k (z-b)^{-1/2} + O(1),\\ A_k = \sin \frac{\pi b}{T} \sum_{j=n+2}^{2n} s_{kj}\Big(c_{0j} + ic_{1j} \cot \frac{\pi b}{T}\Big) \Big/ \sqrt{\frac{\pi}{T} \sin \frac{\pi(b-a)}{T}},\end{gathered} \tag{3.1}$$

where s_{kj} are the elements of the matrix S; $\sqrt{z-b}$ means the single-valued branch in the plane with a cut along the radial $(-\infty, b]$ on the real axis, whose value is equal to 1 for $z - b = 1$.

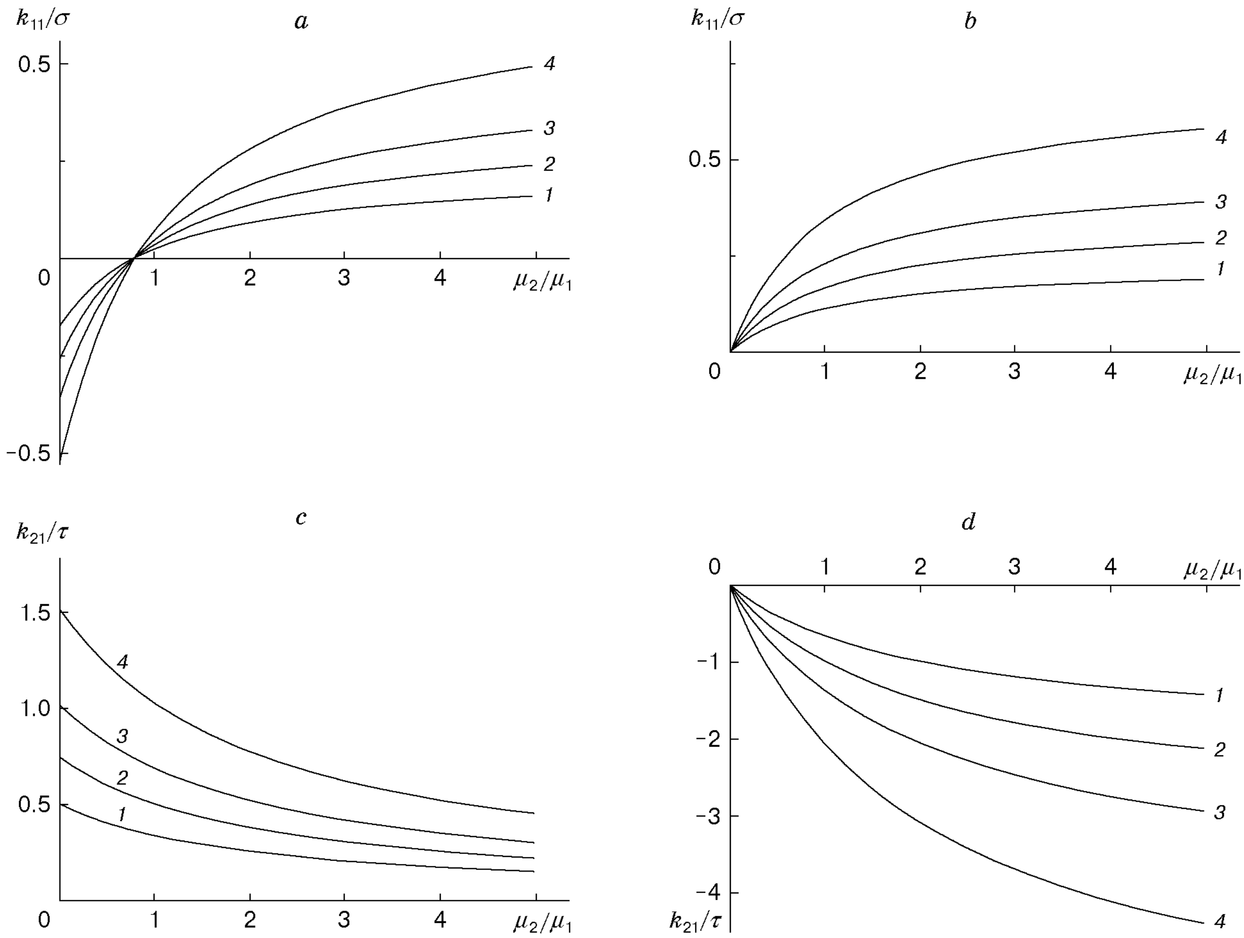

Fig. 1

It follows from (3.1) that in the neighborhood of the joint line, the complex potentials $\Phi_k(z)$ and $\Omega_k(z)$ have the same form as in the neighborhood of a rigid, thin, pointed inclusion [9]. Therefore, near the point $z = b$ in the plate E_k the SIF is given by

$$(k_1 - ik_2)_k(b) = -2æ_k \lim_{z\to b} \sqrt{2\pi(z-b)}\,\Phi_k(z) = -2\sqrt{2\pi}æ_k A_k.$$

Hence, in the neighborhood of the ends of the joint segments, the stress distribution is identical to that near the apex of a rigid, thin, pointed inclusion in a plate.

Example No. 1. Let two plates E_1 and E_2 of equal thickness $h_1 = h_2 = 1$ with elastic characteristics $æ_1 = 2.1$ and $æ_2 = 2.3$ be joined along the segments $[-b + \pi j, b + \pi j]$, where $j = 0, \pm 1, \ldots$. We consider the dependence of the SIF on the ratio μ_2/μ_1 for the following cases.

1. As $y \to +\infty$ and $y \to -\infty$, the stresses acting are, respectively, $(\sigma_y^\infty)_1' = \sigma$ and $(\sigma_x^\infty)_1'' = \sigma(æ_1 - 3)/(1 + æ_1)$ in the plates E_1 and $(\sigma_y^\infty)_2'' = \sigma$ and $(\sigma_x^\infty)_2' = \sigma(æ_2 - 3)/(1 + æ_2)$ in the plates E_2; the remaining stresses and rotations vanish at infinity. In this case, conditions (2.9) are satisfied. Figure 1a shows the coefficient k_{11}/σ versus the ratio μ_2/μ_1 at the points $z = \pm b$ of the plate E_1 for various b. In Fig. 1, curves 1–4 refer to $b = 0.1\pi$, 0.2π, 0.3π, and 0.4π, respectively. The coefficient $k_2 = k_{21}$ for the plate E_1 does not depend on the ratio μ_2/μ_1 for any b, whereas the SIFs for the plate E_2 are obtained by multiplying the corresponding coefficients for the plate E_1 by -1.029. Table 1 lists the coefficient k_{21}/σ for various values of b.

2. As $y \to +\infty$ and $y \to -\infty$, the only nonvanishing stresses in the plate E_1 are $(\sigma_y^\infty)_1' = (\sigma_y^\infty)_1'' = \sigma$. Hence, $k_{21} = 0$ for the plate E_1. Figure 1b shows the coefficient k_{11}/σ versus the ratio μ_2/μ_1 at the points $z = \pm b$ of the plate E_1 for various values of b. The SIFs for the plate E_2 are obtained by multiplying the corresponding coefficients for the plate E_1 by -1.029.

TABLE 1

b	k_{21}/σ	k_{11}/τ
0.1π	-2.307	4.614
0.2π	-1.543	3.086
0.3π	-1.121	2.242
0.4π	-0.750	1.499

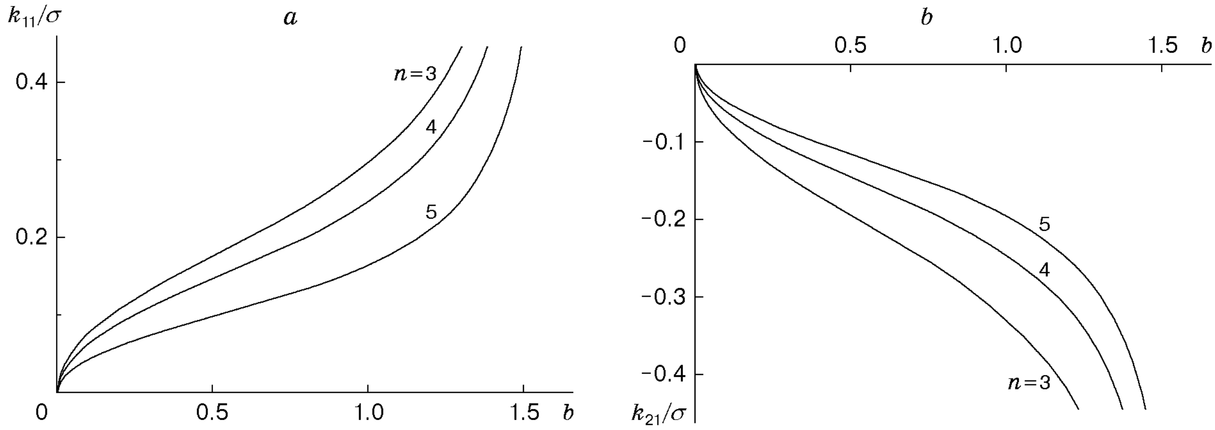

Fig. 2

Example No. 2. Let three plates E_1, E_2, and E_3 of equal unit thickness with elastic characteristics $æ_1 = æ_3 = 2.1$, $æ_2 = 2.3$, and $\mu_1 = \mu_3$ be joined along the segments $[-b + \pi j, b + \pi j]$ $(j = 0, \pm 1, \ldots)$. We now consider the dependence of the SIF on the ratio μ_2/μ_1 for the following cases.

1. As $y \to +\infty$ and $y \to -\infty$, the stresses in the plate E_2 have the form $(\tau_{xy}^\infty)'_2 = (\tau_{xy}^\infty)''_2 = \tau$; the remaining initial data are zero. Now $k_1 = 0$ for all plates. Figure 1c shows the coefficient $k_{21}/\tau = k_{23}/\tau$ versus the ratio μ_2/μ_1 at the points $z = \pm b$ of the plates E_1 and E_3 for various values of b. In this case, the SIFs for the plate E_2 are obtained by multiplying the SIFs for the plate E_1 by -2.058.

2. The shear stresses $(\tau_{xy}^\infty)'_1 = (\tau_{xy}^\infty)''_3 = 2\tau$ and the rotations $(\omega^\infty)''_1 = \tau/\mu_1$ and $(\omega^\infty)'_3 = \tau/\mu_3$ occur in the plate E_1 as $y \to +\infty$ and in the plate E_3 as $y \to -\infty$; the remaining initial data are zero. Hence, the coefficient $k_1 = k_{11}$ of the plate E_1 does not depend on the ratio μ_2/μ_1 for any b (the coefficient k_{11}/τ as a function of b is given in Table 1). Figure 1d shows the coefficient k_{21}/τ versus the ratio μ_2/μ_1 for various values of b. In this case, the SIFs for the plates E_1 and E_3 are the same, and the SIFs for the plate E_2 are obtained by multiplying the SIFs for the plate E_1 by -2.058.

Example No. 3. Let the plates $E_1, E_2, \ldots, E_n$ of equal thickness with equal elastic characteristics $æ$ and μ are joined as a packet along the segments $[-b + \pi j, b + \pi j]$. As $y \to +\infty$ and $y \to -\infty$, the stresses $(\sigma_y^\infty)'_1 = (\sigma_y^\infty)''_n = \sigma$ and $(\sigma_x^\infty)''_1 = (\sigma_x^\infty)'_n = \sigma(æ_1 - 3)/(1 + æ_1)$ act in the plates E_1 and E_n, and the remaining initial data are zero. Then, $k_1 = 0$ for the case of two plates; for the case of three to five plates, Figs. 2a and b show this coefficient versus the coefficient b for the plates E_1 and E_2, respectively.

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REFERENCES

1. V. V. Sil'vestrov and I. A. Ivanov, "Tension of two elastic plates joined along a periodic system of segments," *Izv. Inzh.-Tekhnol. Akad. Chuvash. Resp.*, Nos. 3/4 (1998); Nos. 1/2, 42–46 (1999).
2. V. V. Sil'vestrov and G. E. Chekmarev, "A packet of thin elastic plates joined along collinear segments," in: *Boundary-Value Problems and Their Applications* [in Russian], Izd. Chuvash. Univ., Cheboksary (1992), pp. 38–42.
3. V. V. Sil'vestrov and A. V. Shumilov, "A packet of thin elastic plates joined along concentric circles," *Izv. Inzh.-Tekhnol. Akad. Chuvash. Resp.*, Nos. 1/2, 142–148 (1997).

4. V. V. Sil'vestrov and A. V. Shumilov, "A problem of a packet of elastic plates joined along curves," *Izv. Ross. Akad. Nauk, Mekh. Tverd. Tela*, No. 5, 166–174 (2000).
5. N. I. Muskhelishvili, *Singular Integral Equations*, Noordhoff, Leyden (1977).
6. A. I. Markushevich, *Theory of Analytic Functions* [in Russian], Nauka, Moscow (1968).
7. P. Lancaster, *Theory of Matrices*, Academic Press, New York–London (1969).
8. L. I. Chibrikova, *Basic Boundary-Value Problems for Analytic Functions* [in Russian], Izd. Kazan. Univ., Kazan' (1977).
9. L. T. Berezhnitskii, V. V. Panasyuk, and N. G. Stashchuk, *Interaction of Rigid Linear Inclusions and Cracks in Solids* [in Russian], Naukova Dumka, Kiev (1983).